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On G-invex multiobjective programming. Part II. Duality

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Abstract This paper represents the second part of a study concerning the so-called G-multiobjective programming. A new approach to duality in differentiable vector optimization problems is presented. The techniques used are based on the results established in the paper: *On G-invex multiobjective programming. Part I. Optimality* by T.Antczak. In this work, we use a generalization of convexity, namely G-invexity, to prove new duality results for nonlinear differentiable multiobjective programming problems. For such vector optimization problems, a number of new vector duality problems is introduced. The so-called G-Mond–Weir, G-Wolfe and G-mixed dual vector problems to the primal one are defined. Furthermore, various so-called G-duality theorems are proved between the considered differentiable multiobjective programming problem and its nonconvex vector G-dual problems. Some previous duality results for differentiable multiobjective programming problems turn out to be special cases of the results described in the paper.

Keywords (strictly) G-invex vector function with respect to η · G-Karush–Kuhn–Tucker necessary optimality conditions · G-Mond–Weir vector dual problems · G-Wolfe vector dual problem · G-mixed vector dual problem

1 Introduction

In the classical theory of duality, the theorems on duality in various senses are based on convexity assumptions, which are rather restrictive in applications. Many attempts have been made to weaken these assumptions by introducing various generalized convexity concepts. In [12], Hanson proved the Karush–Kuhn–Tucker sufficient optimality conditions and the Wolfe duality results for a wider class of mathematical programming problems involving functions called invex after the coinage of Craven [7].

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In the recent years, duality in vector optimization has been attracting the interest of many researches. Such optimization problems with several objectives conflicting with one another reflect the complexity of the real world and are encountered in various fields. Many authors have developed the necessary and/or sufficient conditions for (weak) Pareto optimal solutions in multiobjective programming problems (see, for example, [8,11,16,21,22,24], and others)

On the other hand, the duality theory has been another focal issue for a long time, especially in convex multiobjective programming problems (see, for example [6,13,17,22], and others). However, in their most general form, such optimization problems are nonconvex and therefore difficult to analyze. The focus of attention over the last few years has been on the development of new classes of generalized convex functions which are more useful tools to prove duality theorems for such optimization problems. An overview of duality theory for linear and nonlinear cases is presented in Nakayama [20]. In [22], Taninio and Sawaragi studied the duality theory in multiobjective programming on finite dimensional real spaces. They gave a generalization of the definition of a saddle point for the vector-valued Lagrangian function and used the Lagrange multipliers in the dual functional to establish necessary conditions for Pareto optimality in the primal vector optimization problem. Luc [18] studied different vector-valued Lagrangian functions and established duality results for multiobjective programming problems under appropriate convex assumptions. Weir et al. [23] used the Lagrangian description of a weak minimum to establish quasiduality and weak and strong duality theorems in the case where the dual optimization problem has the same objective function as the primal. Egudo and Hanson [10] established some duality results for differentiable multiobjective programming problems with invex functions. In [15], Kaul et al. considered Wolfe type and Mond–Weir type duals and generalized duality results of Weir [25] under weaker invexity assumptions. Craven and Glover [8] proved duality theorems for the so-called cone invex programs. Giorgi and Guerraggio [11] used the introduced broad classes of generalized invex vector functions to prove some duality results for both smooth and nonsmooth multiobjective programming problems. Weir and Jeyakumar [26] considered vector optimization problems in real normed vector spaces and established weak and strong duality theorems for vector optimization problems with cone pre-invex functions. Bector et al. [5] established some duality results for the so-called vector valued B-invex programming problems. In [1], Antczak proved some sufficient optimality conditions and various duality theorems for differentiable multiobjective problems with (p, r)-invex functions. Jeyakumar and Mond [14] introduced the class of the so-called V-invex functions to prove some optimality and duality results for a larger class of differentiable vector optimization problems than problems under invexity assumption. The results established by Jeyakumar and Mond have been generalized by Antczak [2] for a large class of multiobjective programming problems involving V-r-invex functions.

This paper represents the second part of a study concerning the so-called G-multiobjective programming. In this paper, we formulate new various vector dual problems for differentiable nonconvex multiobjective programming problems. In this way, we introduce various vector G-dual problems in the format of Mond–Weir, vector G-dual problem in the sense of Wolfe, and various vector mixed G-dual problems for the considered multiobjective programming problem.

In order to establish various duality theorems for the considered differentiable multiobjective programming problem with vector G-invex functions, we apply new necessary optimality conditions, the so-called G-Karush–Kuhn–Tucker necessary optimality conditions, originally introduced in the first part of our consideration (see [4]).

In this work, we introduce a number of models of the so-called vector G-dual problems in the format of Mond–Weir for the considered multiobjective programming problem. These vector G-dual problems are different from vector duals in the sense of Mond–Weir known in the literature. Various duality results are established by using a Pareto type relation between the primal and dual objective functions in these vector optimization problems and the concept of vectorial G-invexity. Also new G-Karush–Kuhn–Tucker necessary optimality conditions are used to obtain these duality results. Furthermore, two types of converse duality theorems (the so-called G-Mond–Weir converse duality theorems) are established. In this way, we introduce a new type of Mond–Weir converse duality named no-maximal G-converse duality. It is well known in the literature that to prove standard converse duality in the format Mond–Weir it should be assumed that the considered feasible point in Mond–Weir dual problem is its (weak) maximum point. However, the introduced no-maximal G-converse duality between vector G-Mond–Weir dual problems and the considered multiobjective programming problem (VP) can be proved under weaker assumptions. In order to prove it, we assume that the point considered in the no-maximal G-converse duality theorem is only feasible in vector G-Mond–Weir dual problem. But some stronger constraints should be imposed on the functions G_g and G_h, constituting the introduced vector G-Mond–Weir dual problems, to prove this result. What is more, for one of the introduced vector G-Mond–Weir dual problems, the converse duality theorem in the standard form is proved under the assumption that the considered feasible point in G-Mond–Weir dual problem is its (weak) maximum point. Whereas the introduced no-maximal G-converse duality theorem between this vector G-Mond–Weir dual problem and the multiobjective programming problem can be established under weaker assumptions than the standard converse duality known in the literature.

Furthermore, we establish a new Wolfe-type duality for a differentiable multiobjective programming problem with nonconvex functions. For the scalar optimization problem, the duality problem of Wolfe type was introduced by Wolfe [28]. Wolfe type duality for vector optimization problems has been considered, for example, in [1,22,23].

The G-Wolfe duality for multiobjective programming formulated in this paper is new and it is different from the duality of this type known in the literature. The introduced G-Wolfe duality is designed for differentiable vector optimization problems involving nonconvex functions. In order to prove various G-Wolfe duality theorems between the original multiobjective programming problem and its vector G-Wolfe dual problem, we introduce the definition of a so-called vector-valued G-Lagrange function. Our purpose in the G-Wolfe duality is to attempt to solve the primary multiobjective programming problem. In our formulation, it is easier to find the desired solution for some class of nonconvex vector dual problems than in the case of standard vector Wolfe dual problem known in the literature.

Further, a number of vector G-mixed dual problems is introduced for the considered multiobjective programming problem. To formulate these vector dual problems, we use the definition of a G-Lagrange function previously defined for G-Wolfe duality. We also establish various G-mixed dual theorems between the primal multiobjective programming problem and the introduced vector G-mixed dual problems. These introduced vector G-mixed dual problems are also new and different from vector mixed dual problems for vector optimization problems known in the literature.

2 Vector G-invex functions

In this section, we provide some definitions and some results that we shall use in the sequel. The following convention for equalities and inequalities will be used throughout the paper.

For any $x = (x_1, x_2, \ldots, x_n)^T$, $y = (y_1, y_2, \ldots, y_n)^T$, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, \ldots, n$;
(ii) $x < y$ if and only if $x_i < y_i$ for all $i = 1, 2, \ldots, n$;
(iii) $x \leqq y$ if and only if $x_i \leq y_i$ for all $i = 1, 2, \ldots, n$;
(iv) $x \leq y$ if and only if $x \leqq y$ and $x \neq y, n > 1$.

Throughout the paper, we will use the same notation for row and column vectors when the interpretation is obvious.

Definition 1 A function $f : R \to R$ is said to be strictly increasing if and only if

$$\forall x, y \in R \quad x < y \quad \Longrightarrow \quad f(x) < f(y).$$

Let $f = (f_1, \ldots, f_k) : X \to R^k$ be a vector-valued differentiable function defined on a nonempty open set $X \subset R^n$, and let $I_{f_i}(X)$, $i = 1, \ldots, k$, be the range of f_i, that is, the image of X under f_i.

In [4], we introduce a definition of a new class of vector-valued nonconvex functions, the so-called vector G-invex functions. Now, for a reader's convenience, we recall this definition.

Definition 2 Let $f : X \to R^k$ be a vector-valued differentiable function defined on a nonempty open set $X \subset R^n$, and let $u \in X$. We assume that there exists a differentiable vector-valued function $G_f = \left(G_{f_1}, \ldots, G_{f_k}\right) : R \to R^k$ such that any its component $G_{f_i} : I_{f_i}(X) \to R$ is a strictly increasing function on its domain. If, moreover, there exists a vector-valued function $\eta : X \times X \to R^n$ such that, for any $i = 1, \ldots, k$, and all $x \in X$ $(x \neq u)$,

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) - G'_{f_i}(f_i(u)) \nabla f_i(u) \eta(x, u) \geqq 0 \quad (>), \tag{1}$$

then f is said to be a (strictly) vector G_f-invex function at u on X (with respect to η) (or shortly, G_f-invex function at u on X). If (1) is satisfied for each $u \in X$, then f is vector G_f-invex on X with respect to η.
If a function $f_i = 1, \ldots, k$, satisfies (1), we also say that f_i is a G_{f_i}-invex function at u on X with respect to η.

Remark 3 In order to define an analogous class of (strictly) vector G_f-incave functions with respect to η, the direction of the inequality in the definition of these functions should be changed to the opposite one.

Remark 4 In the case when $G_{f_i}(a) \equiv a, i = 1, \ldots, k$, for any $a \in I_{f_i}(X)$, we obtain a definition of a vector-valued invex function.

For some properties of a class of vector G-invex functions, the readers are advised to consult [4] and also, in the scalar case, [3].

3 The G-Karush–Kuhn–Tucker necessary optimality conditions in multiobjective programming

In this paper, we consider the following multiobjective programming problem (VP):

$$\begin{aligned} V\text{-minimize } f(x) &:= (f_1(x), \ldots, f_k(x)) \\ g(x) &\leqq 0, \\ h(x) &= 0, \\ x &\in X, \end{aligned} \tag{VP}$$

where $f_i : X \to R$, $i \in I = \{1, \ldots, k\}$, g_j: $X \to R$, $j \in J = \{1, \ldots, m\}$, h_t: $X \to R$, $t \in T = \{1, \ldots, p\}$ are differentiable functions on a nonempty open set $X \subset R^n$.

Let $D = \{x \in X : g_j(x) \leqq 0, j \in J, h_t(x) = 0, t \in T\}$ be the set of all feasible solutions for problem (VP). Further, we denote by $J(z) := \{j \in J : g_j(z) = 0\}$ the set of inequality constraint functions active at $z \in D$ and by $I(z) := \{i \in I : \lambda_i > 0\}$ the objective functions indices set, for which the corresponding Lagrange multiplier is not equal 0.

For such optimization problems, minimization means in general obtaining (weak) Pareto optimal solutions in the following sense:

Definition 5 A feasible point $\overline{x}$ is said to be a Pareto solution (an efficient solution) for (VP) if and only if there exists no $x \in D$ such that

$$f(x) \leq f(\overline{x}).$$

Definition 6 A feasible point $\overline{x}$ is said to be a weak Pareto solution (a weakly efficient solution, a weak minimum) for (VP) if and only if there exists no $x \in D$ such that

$$f(x) < f(\overline{x}).$$

In [4], we established the so-called G-Karush–Kuhn–Tucker necessary optimality conditions for differentiable multiobjective programming problem under the Kuhn-Tucker constraint qualification. Now, for the reader's convenience we recall the definition of the Kuhn–Tucker constraint qualification and the formulation of these necessary optimality conditions. To establish the necessary optimality conditions for the considered multiobjective programming problem (VP), we need the definition of the Bouligand tangent cone to a set $W \subset R^k$.

Definition 7 Let $W \subset R^k$. The Bouligand tangent cone to W at $\overline{z} \in W$ is the set $C(W, \overline{z})$ of all vectors $q \in R^k$ such that there exist a sequence $\{z_l\}$ in W and a sequence $\{\lambda_l\}$ of strictly positive real numbers such that,

$$\lim_{l\to\infty} z_l = \overline{z}\,, \quad \lim_{l\to\infty} \lambda_l = 0, \quad \lim_{l\to\infty} \frac{z_l - \overline{z}}{\lambda_l} = q.$$

Remark 8 Note that Lin [16] named any Bouligand tangent vector, that is, any vector $q \in C(W, \overline{z})$, a convergence vector for the set W at $\overline{z}$.

Definition 9 The multiobjective programming problem (VP) is said to satisfy the Kuhn–Tucker constraint qualification at $\overline{x} \in D$ if,

$$C(D, \overline{x}) = \left\{d \in R^n : \nabla g_j(\overline{x})\, d \leqq 0,\ j \in J(\overline{x}),\ \nabla h_t(\overline{x})d = 0, t \in T\right\}.$$

Now, we modify slightly the G-Karush–Kuhn–Tucker necessary optimality conditions established in [4]. More exactly, we prove that, if $\overline{x} \in D$ is a (weak) Pareto optimal point, then there exist Lagrange multipliers $\overline{\lambda}_i, i \in I$, associated with the objective functions, satisfy $\sum_{i=1}^{k} \overline{\lambda}_i = 1$.

Theorem 10 (G-Karush–Kuhn–Tucker necessary optimality conditions) *Let $\overline{x} \in D$ be a Pareto optimal point (a weak Pareto optimal point) in problem (VP). Moreover, we assume that g satisfies the Kuhn–Tucker constraint qualification at $\overline{x}$. Then there exist $\overline{\lambda} \in R^k_+$, $\overline{\xi} \in R^m_+$ and $\overline{\mu} \in R^p$ such that*

$$\sum_{i=1}^{k} \overline{\lambda}_i G'_{f_i}\left(f_i(\overline{x})\right) \nabla f_i\left(\overline{x}\right) + \sum_{j=1}^{m} \overline{\xi}_j G'_{g_j}\left(g_j(\overline{x})\right) \nabla g_j\left(\overline{x}\right) + \sum_{t=1}^{p} \overline{\mu}_t G'_{h_t}\left(h_t(\overline{x})\right) \nabla h_t(\overline{x}) = 0, \tag{2}$$

$$\overline{\xi}_j \left[G_{g_j}\left(g_j\left(x\right)\right) - G_{g_j}\left(g_j\left(\overline{x}\right)\right)\right] \leqq 0, \;\; j \in J, \;\; \forall x \in D, \tag{3}$$

$$\overline{\lambda} \geq 0, \; \sum_{i=1}^{k} \overline{\lambda}_i = 1, \overline{\xi} \geqq 0, \tag{4}$$

where G_{f_i}, $i \in I$, are differentiable real-valued strictly increasing functions defined on $I_{f_i}(D)$, G_{g_j}, $j \in J$, are differentiable real-valued strictly increasing functions defined on $I_{g_i}(D)$, and G_{h_t},$t \in T$, are differentiable real-valued strictly increasing functions defined on $I_{h_t}(D)$.

Proof The proof of this theorem follows from the G-Karush–Kuhn–Tucker necessary optimality conditions established in [4]. Indeed, by Theorem 16 [4], it follows that there exist $\widehat{\lambda} \in R^k_+$, $\widehat{\xi} \in R^m_+$ and $\widehat{\mu} \in R^p$ such that the conditions (2)–(3) are fulfilled with these Lagrange multipliers. Therefore, to prove this theorem, it is sufficient only to show that there exist $\overline{\lambda} \in R^k_+$, $\overline{\xi} \in R^m_+$ and $\overline{\mu} \in R^p$ such that $\sum_{i=1}^{k} \overline{\lambda}_i = 1$. We set

$$\overline{\lambda}_q = \frac{\widehat{\lambda}_i}{1 + \sum_{i=1, i \neq q}^{k} \widehat{\lambda}_i} \quad \text{for some } q \in I(\overline{x}), \tag{5}$$

$$\overline{\lambda}_i = \frac{\widehat{\lambda}_i}{1 + \sum_{i=1, i \neq q}^{k} \widehat{\lambda}_i} \quad \text{for } i \in I, i \neq q, \tag{6}$$

$$\overline{\xi}_j = \frac{\widehat{\xi}_j}{1 + \sum_{i=1, i \neq q}^{k} \widehat{\lambda}_i} \quad \text{for } j \in J, \tag{7}$$

$$\overline{\mu}_t = \frac{\widehat{\mu}_t}{1 + \sum_{i=1, i \neq q}^{k} \widehat{\lambda}_i} \quad \text{for } t \in T. \tag{8}$$

It is not difficult to see that the G-Karush–Kuhn–Tucker necessary optimality conditions are satisfied with Lagrange multipliers $\overline{\lambda} \in R^k_+$, $\overline{\xi} \in R^m_+$ and $\overline{\mu} \in R^p$ satisfying (5)–(8). □

4 G-Mond–Weir vector duality

Now, in relation to (VP), we consider the following multiobjective dual problem, which is in the format of Mond–Weir [19]

$$f(y) = (f_1(y), f_2(y), \ldots, f_k(y)) \to \max$$
$$\left[\sum_{i=1}^{k} \lambda_i G'_{f_i}(f_i(y)) \nabla f_i(y) + \sum_{j=1}^{m} \xi_j G'_{g_i}\left(g_j(y)\right) \nabla g_j(y) + \sum_{t=1}^{p} \mu_t G'_{h_t}(h_t(y)) \nabla h_t(y)\right] \eta(x,y) \geqq 0, \ \forall x \in D, \quad (G\text{-VMWD1})$$
$$\sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(y)\right) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(y)) \geqq 0,$$
$$y \in X,$$
$$\lambda \in R^k, \ \lambda \geq 0, \ \lambda \mathbf{e} = 1,$$
$$\xi \in R^m, \ \xi \geqq 0,$$
$$\mu \in R^q,$$

where $e = (1, \ldots, 1) \in R^k$, G_{f_i}, $i \in I$, are differentiable real-valued strictly increasing functions defined on $I_{f_i}(X)$, G_{g_j}, $j \in J$, are differentiable real-valued strictly increasing functions defined on $I_{g_j}(X)$, and G_{h_t}, $t \in T$, are differentiable real-valued strictly increasing functions defined on $I_{h_t}(X)$.
We call (G-VMWD1) the G-Mond–Weir vector dual problem for the multiobjective programming problem (VP).

Let W_1 denote the set of all feasible points of (G-VMWD1) and $pr_X W_1$ be the projection of the set W_1 on X, that is, $pr_X W_1 := \{y \in X : (y, \lambda, \xi, \mu) \in W_1\}$. Moreover, for a given $(y, \lambda, \xi, \mu) \in W_1$, we denote by $I(y)$ the set of objective functions indices for which a corresponding Lagrange multiplier is positive, that is, $I(y) := \{i \in I : \lambda_i > 0\}$, and, moreover, we denote by $T^+(y)$ and $T^-(y)$ the sets of equality constraints indices for which the corresponding Lagrange multiplier is positive and negative, respectively, that is, $T^+(y) = \{t \in T : \mu_t > 0\}$ and $T^-(y) = \{t \in T : \mu_t < 0\}$.

The following lemma which will be used in the sequel, is an immediate consequence of the introduced definitions of vector-valued G-invex functions and, therefore, its simple proof will be omitted.

Lemma 11 *Let (y, λ, ξ, μ) be any feasible solution in (G-VMWD1). If g is G_g-invex with respect to η at $y \in pr_X W_1$ on $D \cup pr_X W_1$, h_t, $t \in T^+(y)$, is G_{h_t}-invex with respect to η at $y \in pr_X W_1$ on $D \cup pr_X W_1$, h_t, $t \in T^-(y)$, is G_{h_t}-incave with respect to η at $y \in pr_X W_1$ on $D \cup pr_X W_1$, $G_{g_j}(0) = 0$ for $j \in J$ and $G_{h_t}(0) = 0$ for $t \in T^+(y) \cup T^-(y)$, then*

$$\left[\sum_{j=1}^{m} \xi_j G'_{g_j}\left(g_j(y)\right) \nabla g_j(y) + \sum_{t=1}^{p} \mu_t G'_{h_t}(h_t(y)) \nabla h_t(y)\right] \eta(x,y) \leqq 0, \quad \forall x \in D. \quad (9)$$

Theorem 12 (G-weak duality) *Consider the multiobjective problems (VP) and (G-VMWD1). Let x and (y, λ, ξ, μ) be any feasible solutions for (VP) and (G-VMWD1), respectively. Further, we assume that f is G_f-invex with respect to η at $y \in pr_X W_1$ on $D \cup pr_X W_1$, g is G_g-invex with respect to η at $y \in pr_X W_1$ on $D \cup pr_X W_1$, h_t, $t \in T^+(y)$, is G_{h_t}-invex with respect to η at $y \in pr_X W_1$ on $D \cup pr_X W_1$, h_t, $t \in T^-(y)$, is G_{h_t}-incave with respect to η at $y \in pr_X W_1$ on $D \cup pr_X W_1$, $G_{g_j}(0) = 0$ for $j \in J$ and $G_{h_t}(0) = 0$ for $t \in T^+(y) \cup T^-(y)$. Then $f(x) \nless f(y)$.*

Proof Let x and (y, λ, ξ, μ) be feasible solutions for (VP) and (G-VMWD1), respectively. We proceed by contradiction. Suppose that $f(x) < f(y)$. By assumption, f is G_f-invex with respect to η at y on $D \cup pr_X W_1$. Then, by Definition 2, for any $i \in I$,

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(y)) - G'_{f_i}(f_i(y)) \nabla f_i(y)\eta(x,y) \geqq 0. \tag{10}$$

Hence, by $f(x) < f(y)$, it follows that,

$$f_i(x) < f_i(y), \quad i \in I. \tag{11}$$

From Definition 2, it follows that $G_{f_i}, i \in I$, are strictly increasing functions on $D \cup pr_X W_1$. Thus, (11) gives

$$G_{f_i}(f_i(x)) < G_{f_i}(f_i(y)), \quad i \in I. \tag{12}$$

Hence, by (10) and (12), we get

$$G'_{f_i}(f_i(y)) \nabla f_i(u)\eta(x,y) < 0, \quad i \in I.$$

Since (y, λ, ξ, μ) is feasible in (G-VMWD1), from the constraints of this dual problem it follows that

$$\sum_{i=1}^{k} \lambda_i G'_{f_i}(f_i(y)) \nabla f_i(y)\eta(x,y) < 0. \tag{13}$$

By assumption, g is G_g-invex with respect to η at $y \in pr_X W_1$ on $D \cup pr_X W_1$, $h_t, t \in T^+(y)$, is G_{h_t}-invex with respect to η at $y \in pr_X W_1$ on $D \cup pr_X W_1$, $h_t, t \in T^-(y)$, is G_{h_t}-incave with respect to η at $y \in pr_X W_1$ on $D \cup pr_X W_1$. Then, by Lemma 11, we have,

$$\left[\sum_{j=1}^{m} \xi_j G'_{g_j}\left(g_j(y)\right) \nabla g_j(y) + \sum_{t=1}^{p} \mu_t G'_{h_t}(h_t(y)) \nabla h_t(y)\right] \eta(x,y) \leqq 0. \tag{14}$$

Adding both sides of (13) and (14), we get the following inequality

$$\left[\sum_{i=1}^{k} \lambda_i G'_{f_i}(f_i(y)) \nabla f_i(y) + \sum_{j=1}^{m} \xi_j G'_{g_j}\left(g_j(y)\right) \nabla g_j(y) + \sum_{t=1}^{p} \mu_t G'_{h_t}(h_t(y)) \nabla h_t(y)\right] \eta(x,y) < 0,$$

which contradicts the first constraint of (G-VMWD1). Thus, the conclusion of theorem is proved. □

Theorem 13 (G-weak duality) *Consider the multiobjective problems (VP) and (G-VMWD1). Let x and (y, λ, ξ, μ) be any feasible solutions for (VP) and (G-VMWD1), respectively. Further, we assume that f is strictly G_f-invex with respect to η at $y \in pr_X W_1$ on $D \cup pr_X W_1$, g is G_g-invex with respect to η at $y \in pr_X W_1$ on $D \cup pr_X W_1$, h_t, $t \in T^+(y)$, is G_{h_t}-invex with respect to η at y on $D \cup pr_X W_1$, h_t, $t \in T^-(y)$, is G_{h_t}-incave with respect to η at $y \in pr_X W_1$ on $D \cup pr_X W_1$, $G_{g_j}(0) = 0$ for $j \in J$ and $G_{h_t}(0) = 0$ for $t \in T^+(y) \cup T^-(y)$. Then $f(x) \nleq f(y)$.*

Theorem 14 (G-strong duality) *Let $\overline{x} \in D$ be a (weak) Pareto solution in (VP) and the Kuhn–Tucker constraint qualification be satisfied at $\overline{x}$. Assume that $G_{h_t}(0) = 0$ for $t \in T^+(\overline{x}) \cup T^-(\overline{x})$. Then there exist $\overline{\lambda} \in R^k_+$, $\overline{\xi} \in R^m_+$, $\overline{\mu} \in R^q$, $\left(\overline{\lambda} \geq 0,\ \overline{\xi} \geqq 0\right)$ $\overline{\lambda} > 0$, $\overline{\xi} \geqq 0$, such that $\left(\overline{x}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ is feasible for (G-VMWD1). If also G-weak duality (Theorem 12* *or Theorem 13) holds, then* $(\overline{x}, \overline{\lambda}, \overline{\xi}, \overline{\mu})$ *is a (weak) maximum for (G-VMWD1), and the objective functions values are equal in problems (VP) and (G-VMWD1).*

Proof By assumption, $\overline{x}$ is a (weak) Pareto solution in (VP). Then, there exist $\overline{\lambda} \in R^k_+$, $\overline{\xi} \in R^m_+, \overline{\mu} \in R^q$ $(\overline{\lambda} \geq 0, \overline{\xi} \geqq 0)$ $\overline{\lambda} > 0, \overline{\xi} \geqq 0$, such that the G-Karush–Kuhn–Tucker conditions (2)–(4) hold. Thus, the G-Karush–Kuhn–Tucker condition (3)–(4) imply the inequality

$$\sum_{j=1}^{m} \overline{\xi}_j G_{g_j} \left(g_j\left(\overline{x}\right)\right) \geqq 0. \tag{15}$$

By assumption, $G_{h_t}(0) = 0$ for $t \in T^+(\overline{x}) \cup T^-(\overline{x})$. Since $\overline{x} \in D$, then

$$\sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\overline{x})\right) \geqq 0. \tag{16}$$

Using the G-Karush–Kuhn–Tucker conditions (2) together with (15) and (16), we obtain the feasibility of $(\overline{x}, \overline{\lambda}, \overline{\xi}, \overline{\mu})$ in (G-VMWD1). Also, by weak duality (Theorem 12 or Theorem 13), it follows that $(\overline{x}, \overline{\lambda}, \overline{\xi}, \overline{\mu})$ is a (weak) maximum in (G-VMWD1). □

Theorem 15 (G-converse duality) *Let* $(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu})$ *be a weak maximum (maximum) in (G-VMWD1) and* $\overline{y} \in D$. *Further, assume that* f *is (strictly)* G_f*-invex with respect to* η *at* $\overline{y}$ *on* $D \cup pr_X W_1$, g *is* G_g*-invex with respect to* η *at* $\overline{y}$ *on* $D \cup pr_X W_1$, h_t, $t \in T^+(\overline{y})$, *is* G_{h_t}*-invex with respect to* η *at* $\overline{y}$ *on* $D \cup pr_X W_1$ *and* h_t, $t \in T^-(\overline{y})$, *is* G_{h_t}*-incave with respect to* η *at* $\overline{y}$ *on* $D \cup pr_X W_1$. *Then* $\overline{y}$ *is a weak Pareto optimal (Pareto optimal) in (VP).*

Proof Let $(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu})$ be a weak maximum in (G-VMWD1) such that $\overline{y} \in D$. By means of contradiction, we suppose that there exists $\widetilde{x} \in D$ such that

$$f\left(\widetilde{x}\right) < f\left(\overline{y}\right).$$

By assumption, f is G_f-invex with respect to η at $\overline{y}$ on $D \cup pr_X W_1$. Then, from the definition of G_f-invexity, it follows that G_{f_i}, $i \in I$, are strictly increasing functions on $I_{f_i}(X)$. Thus

$$G_{f_i}\left(f_i\left(\widetilde{x}\right)\right) < G_{f_i}\left(f_i\left(\overline{y}\right)\right), \quad i \in I. \tag{17}$$

Since $\overline{\lambda} \geq 0$, then (17) gives

$$\sum_{i=1}^{k} \overline{\lambda}_i G_{f_i}\left(f(\widetilde{x})\right) < \sum_{i=1}^{k} \overline{\lambda}_i G_{f_i}\left(f(\overline{y})\right). \tag{18}$$

By assumption, $(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu})$ is a weak maximum in (G-VMWD1). Then, by the G-Karush–Kuhn–Tucker necessary optimality condition (3), it follows that

$$\sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j\left(\widetilde{x}\right)\right) \leqq \sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j\left(\overline{y}\right)\right). \tag{19}$$

Since $\overline{y} \in D$ and $\widetilde{x} \in D$, then

$$\sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\widetilde{x})\right) - \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\overline{y})\right) = 0. \tag{20}$$

By assumption, f is G_f-invex with respect to η at $\overline{y}$ on $D \cup pr_X W_1$, g is G_g-invex with respect to η at $\overline{y}$ on $D \cup pr_X W_1$ and h_t, $t \in T^+(\overline{y})$, is G_{h_t}-invex with respect to η at $\overline{y}$ on

$D \cup pr_X W_1$, h_t, $t \in T^-(\overline{y})$, is G_{h_t}-incave with respect to η at $\overline{y}$ on $D \cup pr_X W_1$. Then, by Definition 2, we have, respectively,

$$G_{f_i}(f_i(\widetilde{x})) - G_{f_i}(f_i(y)) - G'_{f_i}(f_i(\overline{y}))\nabla f_i(\overline{y})\eta(\widetilde{x},\overline{y}) \geqq 0, \quad i \in I,$$

$$G_{g_j}\left(g_j(\widetilde{x})\right) - G_{g_j}\left(g_j(\overline{y})\right) \geqq G'_{g_j}\left(g_j(\overline{y})\right)\nabla g_j(\overline{y})\,\eta(\widetilde{x},\overline{y}), \quad j \in J,$$

$$G_{h_t}(h_t(\widetilde{x})) - G_{h_t}(h_t(\overline{y})) \geqq G'_{h_t}(h_t(\overline{y}))\nabla h_t(\overline{y})\eta(\widetilde{x},\overline{y}), \quad t \in T^+(\overline{y}),$$

$$G_{h_t}(h_t(\widetilde{x})) - G_{h_t}(h_t(\overline{y})) \leqq G'_{h_t}(h_t(\overline{y}))\nabla h_t(\overline{y})\eta(\widetilde{x},\overline{y}), \quad t \in T^-(\overline{y}).$$

From the feasibility of $\left(\overline{y},\overline{\lambda},\overline{\xi},\overline{\mu}\right)$ in (G-VMWD1) follows

$$\sum_{i=1}^{k}\overline{\lambda}_i G_{f_i}(f_i(\widetilde{x})) - \sum_{i=1}^{k}\overline{\lambda}_i G_{f_i}(f_i(\overline{y})) \geqq \sum_{i=1}^{k}\overline{\lambda}_i G'_{f_i}(f_i(\overline{y}))\nabla f_i(\overline{y})\eta(\widetilde{x},\overline{y}) \tag{21}$$

$$\sum_{j=1}^{m}\overline{\xi}_j G_{g_j}\left(g_j(\widetilde{x})\right) - \sum_{j=1}^{m}\overline{\xi}_j G_{g_j}\left(g_j(\overline{y})\right) \geqq \sum_{j=1}^{m}\overline{\xi}_j G'_{g_j}\left(g_j(\overline{y})\right)\nabla g_j(\overline{y})\,\eta(\widetilde{x},\overline{y}), \tag{22}$$

$$\sum_{t=1}^{p}\overline{\mu}_t G_{h_t}(h_t(\widetilde{x})) - \sum_{t=1}^{p}\overline{\mu}_t G_{h_t}(h_t(\overline{y})) \geqq \sum_{t=1}^{p}\overline{\mu}_t G'_{h_t}(h_t(\overline{y}))\nabla h_t(\overline{y})\eta(\widetilde{x},\overline{y}). \tag{23}$$

Thus, using (18–20) together with (21–23), we get, respectively,

$$\sum_{i=1}^{k}\overline{\lambda}_i G'_{f_i}(f_i(\overline{y}))\nabla f_i(\overline{y})\eta(\widetilde{x},\overline{y}) < 0, \tag{24}$$

$$\sum_{j=1}^{m}\overline{\xi}_j G'_{g_j}\left(g_j(\overline{y})\right)\nabla g_j(\overline{y})\,\eta(\widetilde{x},\overline{y}) \leqq 0, \tag{25}$$

$$\sum_{t=1}^{p}\overline{\mu}_t G'_{h_t}(h_t(\overline{y}))\nabla h_t(\overline{y})\eta(\widetilde{x},\overline{y}) \leqq 0. \tag{26}$$

Adding both sides of (24)–(26), we obtain the following inequality

$$\left[\sum_{i=1}^{k}\overline{\lambda}_i G'_{f_i}(f_i(\overline{y}))\nabla f_i(\overline{y}) + \sum_{j=1}^{m}\overline{\xi}_j G'_{g_j}\left(g_j(\overline{y})\right)\nabla g_j(\overline{y}) + \sum_{t=1}^{p}\overline{\mu}_t G'_{h_t}(h_t(\overline{y}))\nabla h_t(\overline{y})\right]\eta(\widetilde{x},\overline{y}) < 0, \tag{27}$$

contradicting the feasibility of $\left(\overline{y},\overline{\lambda},\overline{\xi},\overline{\mu}\right)$ in (G-VMWD1).

Proof for $\overline{y}$ to be Pareto optimal in (VP) is similar, but f has to be assumed strictly G_f-invex with respect to η at $\overline{y}$ on $D \cup pr_X W_1$. □

Remark 16 As follows from the proof of the G-converse duality theorem in the standard form (Theorem 15), the assumption that $\left(\overline{y},\overline{\lambda},\overline{\xi},\overline{\mu}\right)$ is a (weak) maximum point in the vector dual problem (G-VMWD1) can be weakened. Indeed, it is sufficient to assume that $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ is only a G-Karush–Kuhn–Tucker point.

Now, we introduce a new kind of converse duality between the multiobjective programming problem (VP) and its vector Mond–Weir dual problem (G-VMWD1). It turns out that the G-converse duality theorem between multiobjective problems (VP) and (G-VMWD1) can be proved without assuming that the point $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$, being feasible in ($G$-VMWD1), is a (weak) maximum in G-Mond–Weir dual problem (G-VMWD1) and also without assuming that it is a G-Karush–Kuhn–Tucker point. But, in this case, some stronger assumptions have to imposed on the functions G_{g_j}, $j \in J$, and G_{h_t}, $t \in T$.

Theorem 17 (No-maximal G-converse duality) *Let $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ be a feasible solution in (G-VMWD1) such that $\overline{y} \in D$. Further, assume that f is (strictly) G_f-invex with respect to η at $\overline{y}$ on $D \cup pr_X W_1$, g is G_g-invex with respect to η at $\overline{y}$ on $D \cup pr_X W_1$, h_t, $t \in T^+(\overline{y})$, is G_{h_t}-invex with respect to η at $\overline{y}$ on $D \cup pr_X W_1$ and h_t, $t \in T^-(\overline{y})$, is G_{h_t}-incave with respect to η at $\overline{y}$ on $D \cup pr_X W_1$, $G_{g_j}(0) = 0$ for $j \in J$ and $G_{h_t}(0) = 0$ for $t \in T^+(\overline{y}) \cup T^-(\overline{y})$. Then $\overline{y}$ is a weak Pareto optimal (Pareto optimal) solution in (VP).*

Proof Let $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ be feasible in (G-VMWD1). Suppose, contrary to the result, that $\overline{y}$ is not a weak Pareto optimal solution in (VP). Then there exists $\widetilde{x} \in D$ such that

$$f(\widetilde{x}) < f(\overline{y}).$$

In the similar manner, as in proof of Theorem 15, it can be proved that the inequality (24) is satisfied. By assumption, f is G_f-invex with respect to η at $\overline{y}$ on $D \cup pr_X W_1$, G_g-invex with respect to η at $\overline{y}$ on $D \cup pr_X W_1$, h_t, $t \in T^+(\overline{y})$, is G_{h_t}-invex with respect to η at $\overline{y}$ on $D \cup pr_X W_1$ and h_t, $t \in T^-(\overline{y})$, is G_{h_t}-incave with respect to η at $\overline{y}$ on $D \cup pr_X W_1$. Since $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ is feasible in (G-VMWD1), then the inequalities (21)–(23) are also fulfilled. Adding both sides of (22) and (23), we get

$$\begin{aligned}&\sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\widetilde{x})\right) + \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\widetilde{x})\right) - \left[\sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\overline{y})\right) + \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\overline{y})\right)\right] \\ &\quad \geqq \left[\sum_{j=1}^{m} \overline{\xi}_j G'_{g_j}\left(g_j(\overline{y})\right) \nabla g_j(\overline{y}) + \sum_{t=1}^{p} \overline{\mu}_t G'_{h_t}\left(h_t(\overline{y})\right) \nabla h_t(\overline{y})\right] \eta(\widetilde{x}, \overline{y}). \qquad (28)\end{aligned}$$

From Definition 2 follows that G_{g_j}, $j \in J$, is strictly increasing on $I_{g_j}(X)$ and G_{h_t}, $t \in T$, is strictly increasing on $I_{h_t}(X)$. Using $G_{g_j}(0) = 0$ for $j \in J$ and $G_{h_t}(0) = 0$ for $t \in T^+(\overline{y}) \cup T^-(\overline{y})$ together with $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right) \in W$, we obtain

$$\sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\widetilde{x})\right) + \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\widetilde{x})\right) \leqq 0. \qquad (29)$$

Hence, from the feasibility of $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ in (G-VMWD1) and (29), it follows that

$$\sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\widetilde{x})\right) + \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\widetilde{x})\right) - \left[\sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\overline{y})\right) + \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\overline{y})\right)\right] \leqq 0. \qquad (30)$$

By (28) and (30),

$$\left[\sum_{j=1}^{m}\overline{\xi}_j G'_{g_j}\left(g_j(\overline{y})\right)\nabla g_j\left(\overline{y}\right)+\sum_{t=1}^{p}\overline{\mu}_t G'_{h_t}\left(h_t(\overline{y})\right)\nabla h_t(\overline{y})\right]\eta(\widetilde{x},\overline{y})\leqq 0. \tag{31}$$

Adding both sides of (24) and (31), we obtain the inequality (27), which contradicts the feasibility of $\left(\overline{y},\overline{\lambda},\overline{\xi},\overline{\mu}\right)$ in (G-VMWD1). □

Remark 18 Note that, in the case of Pareto optimality, these two G-converse duality theorems can be proved under another restrictions imposed of the functions G_f and G_g. Indeed, in place of strictly G_f-invexity assumption of the objective function f, it can be assumed that at least one of the functions g_j, $j\in J(\overline{x})$, is strictly G_{g_j}-invex, to prove that $\overline{y}$ is a Pareto optimal solution in (VP).

Theorem 19 (G-restricted converse duality) *Let $\overline{x}$ and $\left(\overline{y},\overline{\lambda},\overline{\xi},\overline{\mu}\right)$ be feasible solutions in (VP) and (G-VMWD1), respectively, such that, for any $i\in I$,*

$$G_{f_i}\left(f_i(\overline{x})\right)=G_{f_i}\left(f_i(\overline{y})\right).$$

Moreover, assume that f is G_f-invex with respect to η at $\overline{y}$ on $D\cup pr_X W_1$, g is G_g-invex with respect to η at $\overline{y}$ on $D\cup pr_X W_1$, h_t, $t\in T^+(\overline{y})$, is G_{h_t}-invex with respect to η at $\overline{y}$ on $D\cup pr_X W_1$ and h_t, $t\in T^-(\overline{y})$, is G_{h_t}-incave with respect to η at $\overline{y}$ on $D\cup pr_X W_1$, $G_{g_j}(0)=0$ for $j\in J$ and $G_{h_t}(0)=0$ for $t\in T^+(\overline{y})\cup T^-(\overline{y})$. Then $\overline{x}$ is weak Pareto optimal in (VP) and $\left(\overline{y},\overline{\lambda},\overline{\xi},\overline{\mu}\right)$ is a weak maximum in (G-VMWD1).

Proof This follows on the line of the proof of Theorem 12. □

Theorem 20 (G-restricted converse duality) *Let $\overline{x}$ and $\left(\overline{y},\overline{\lambda},\overline{\xi},\overline{\mu}\right)$ be feasible solutions in (VP) and (G-VMWD1), respectively, such that*

$$G_{f_i}\left(f_i(\overline{x})\right)=G_{f_i}\left(f_i(\overline{y})\right).$$

Moreover, assume that f is strictly G_f-invex with respect to η at $\overline{y}$ on $D\cup pr_X W_1$, g is G_g-invex with respect to η at $\overline{y}$ on $D\cup pr_X W_1$, h_t, $t\in T^+(\overline{y})$, is G_{h_t}-invex with respect to η at $\overline{y}$ on $D\cup pr_X W_1$ and h_t, $t\in T^-(\overline{y})$, is G_{h_t}-incave with respect to η at $\overline{y}$ on $D\cup pr_X W_1$, $G_{g_j}(0)=0$ for $j\in J$ and $G_{h_t}(0)=0$ for $t\in T^+(\overline{y})\cup T^-(\overline{y})$. Then $\overline{x}$ is Pareto optimal in (VP) and $\left(\overline{y},\overline{\lambda},\overline{\xi},\overline{\mu}\right)$ is maximum in (G-VMWD1).

Proof This follows on the line of the proof of Theorem 13. □

Now, for the considered multiobjective programming problem (VP), we introduce a new vector dual problem, which is also in the format of Mond–Weir. In this way, in relation to (VP), we consider the following vector dual problem

$$
\begin{aligned}
&f(y) = (f_1(y), f_2(y), \ldots, f_k(y)) \to \max \\
&\left[\sum_{i=1}^{k} \lambda_i G'_{f_i}(f_i(y)) \nabla f_i(y) + \sum_{j=1}^{m} \xi_j G'_{g_j}\left(g_j(y)\right) \nabla g_j(y) \right. \\
&\left. + \sum_{t=1}^{p} \mu_t G'_{h_t}(h_t(y)) \nabla h_t(y) \right] \eta(x, y) \geqq 0, \quad \forall x \in D, \\
&\sum_{j=1}^{m} \xi_j \left[G_{g_j}\left(g_j(y)\right) - G_{g_j}(g_i(x))\right] \geqq 0, \quad \forall x \in D, \qquad (G\text{-VMWD2}) \\
&\sum_{t=1}^{p} \mu_t \left[G_{h_t}(h_t(y)) - G_{h_t}(h_t(x))\right] \geqq 0, \quad \forall x \in D, \\
&y \in X, \\
&\lambda \in R^k, \ \lambda \geq 0, \ \lambda \mathbf{e} = 1, \\
&\xi \in R^m, \ \xi \geqq 0, \\
&\mu \in R^q,
\end{aligned}
$$

where $\mathbf{e} = (1, \ldots, 1) \in R^k$, and functions G_{f_i}, $i \in I$, G_{g_j}, $j \in J$, G_{h_t}, $t \in T$, are defined in the similar manner as in the case of vector G-Mond–Weir dual problem (G-VMWD1). We will call (G-VMWD2) the vector G-Mond–Weir dual problem with modified constraints.

Let W_2 denote the set of all feasible solutions of (G-VMWD2) and $pr_X W_2$ be the projection of the set W_2 on X, that is, $pr_X W_2 := \{y \in X : (y, \lambda, \xi, \mu) \in W_2\}$.

It turns out that various duality theorems between the considered multiobjective programming problem (VP) and the introduced vector Mond–Weir dual problem with modified constraints (G-VMWD2) can be established under weaker assumptions imposed on functions G_{g_i}, $j \in J$ and G_{h_t}, $t \in T$ than in the case of duality results proved between problems (VP) and (G-VMWD1).

Theorem 21 (G-weak duality) *Consider the multiobjective problems (VP) and (G-VMWD2). Let x and (y, λ, ξ, μ) be any feasible points for (VP) and (G-VMWD2), respectively. Further, we assume that f is G_f-invex with respect to η at $y \in pr_X W_2$ on $D \cup pr_X W_2$, g is G_g-invex with respect to η at $y \in pr_X W_2$ on $D \cup pr_X W_2$, h_t, $t \in T^+(y)$, is G_{h_t}-invex with respect to η at $y \in pr_X W_2$ on $D \cup pr_X W_2$, and h_t, $t \in T^-(y)$, is G_{h_t}-incave with respect to η at $y \in pr_X W_2$ on $D \cup pr_X W_2$. Then $f(x) \nless f(y)$.*

Proof Let x and (y, λ, ξ, μ) be any feasible points for (VP) and (G-VMWD2), respectively. We proceed by contradiction. We assume that $f(x) < f(y)$ and exhibit a contradiction. By assumption, f is G_f-invex with respect to η at y on $D \cup pr_X W_2$. Then, by Definition 2, for any $i \in I$,

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(y)) - G'_{f_i}(f_i(y)) \nabla f_i(y) \eta(x, y) \geqq 0. \tag{32}$$

Thus, by $f(x) < f(y)$, we have

$$f_i(x) < f_i(y), \quad i \in I. \tag{33}$$

From Definition 2 follows that G_{f_i} is a strictly increasing function on $D \cup pr_X W_2$. Thus, (33) gives

$$G_{f_i}(f_i(x)) < G_{f_i}(f_i(y)), \quad i \in I. \tag{34}$$

Hence, by (32) and (34), we get

$$G'_{f_i}\left(f_i(y)\right)\nabla f_i(y)\eta(x,y) < 0, \quad i \in I.$$

Since (y, λ, ξ, μ) is feasible in (G-VMWD2) then, using the constraints of this vector dual problem, we obtain

$$\sum_{i=1}^{k} \lambda_i G'_{f_i}\left(f_i(y)\right)\nabla f_i(y)\eta(x,y) < 0. \tag{35}$$

By assumption, g is G_g-invex with respect to η at $y \in pr_X W_2$ on $D \cup pr_X W_2$, $h_t, t \in T^+(y)$, is G_{h_t}-invex with respect to η at $y \in pr_X W_2$ on $D \cup pr_X W_2$, and $h_t, t \in T^-(y)$, is G_{h_t}-incave with respect to η at $y \in pr_X W_2$ on $D \cup pr_X W_2$. Then, by Definition 2, we have, respectively,

$$\sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(x)\right) - \sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(y)\right) \geqq \sum_{j=1}^{m} \xi_j G'_{g_j}\left(g_j(y)\right)\nabla g_j(y)\,\eta(x,y),$$

$$\sum_{t=1}^{p} \mu_t G_{h_t}\left(h_t(x)\right) - \sum_{t=1}^{p} \mu_t G_{h_t}\left(h_t(y)\right) \geqq \sum_{t=1}^{p} \mu_t G'_{h_t}\left(h_t(y)\right)\nabla h_t(y)\eta(x,y).$$

Since $x \in D$ and $(y, \lambda, \xi, \mu) \in W_2$, then the inequalities above yield, respectively,

$$\sum_{j=1}^{m} \xi_j G'_{g_j}\left(g_j(y)\right)\nabla g_j(y)\,\eta(x,y) \leqq 0,$$

$$\sum_{t=1}^{p} \mu_t G'_{h_t}\left(h_t(y)\right)\nabla h_t(y)\eta(x,y) \leqq 0.$$

Adding both sides of the inequalities above and (35), we obtain

$$\left[\sum_{i=1}^{k} \lambda_i G'_{f_i}\left(f_i(y)\right)\nabla f_i(y) + \sum_{j=1}^{m} \xi_j G'_{g_j}\left(g_j(y)\right)\nabla g_j(y) + \sum_{t=1}^{p} \mu_t G'_{h_t}\left(h_t(y)\right)\nabla h_t(y)\right]\eta(x,y) < 0,$$

which contradicts the first constraint of (G-VMWD2). Thus, the conclusion of theorem is established. □

Theorem 22 (G-strong duality) *Let $\overline{x} \in D$ be a (weak) Pareto solution in (VP) and the Kuhn–Tucker constraint qualification be satisfied at $\overline{x}$. Then there exist $\overline{\lambda} \in R^k_+$, $\overline{\xi} \in R^m_+$, $\overline{\mu} \in R^q$, $\left(\overline{\lambda} \geq 0, \overline{\xi} \geqq 0\right)$ $\overline{\lambda} > 0$, $\overline{\xi} \geqq 0$, such that $\left(\overline{x}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ is feasible for (G-VMWD2). If also G-weak duality (Theorem 21) holds then $\left(\overline{x}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ is a (weak) maximum for (G-VMWD2), and the objective function values are equal in problems (VP) and (G-VMWD2).*

Remark 23 Note that we have established the G-strong duality theorem between the considered multiobjective programming problem (VP) and its vector dual problem (G-VMWD2) under weaker assumption imposed on the functions G_{g_j}, $j \in J(\overline{x})$ and G_{h_t}, $t \in T^+(\overline{x}) \cup T^+(\overline{x})$ than in the case of G-strong duality between problems (VP) and (G-VMWD1).

This also follows from the fact that G-weak duality theorem between problems (VP) and (G-VMWD2) has been proved under weaker constraints imposed on the functions G_{g_j} and G_{h_t} than in the case of G-weak duality between problems (VP) and (G-VMWD1).

Analogously, as in the case of vector G-Mond–Weir dual problem (G-VMWD1), we prove a new kind of converse duality between the multiobjective programming problem (VP) and its vector G-Mond–Weir dual problem (G-VMWD2). We also called it no-maximal G-converse duality in the format of Mond–Weir (with modified constraints).

Theorem 24 (No-maximal G-converse duality) *Let $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ be feasible (G-VMWD2) and $\overline{y} \in D$. Further, assume that f is (strictly) G_f-invex with respect to η at $\overline{y}$ on $D \cup pr_X W_2$, g is G_g-invex with respect to η at $\overline{y}$ on $D \cup pr_X W_2$, h_t, $t \in T^+(\overline{y})$, is G_{h_t}-invex with respect to η at $\overline{y}$ on $D \cup pr_X W_2$, and h_t, $t \in T^-(\overline{y})$, is G_{h_t}-incave with respect to η at $\overline{y}$ on $D \cup pr_X W_2$. Then $\overline{y}$ is a weak Pareto optimal (Pareto optimal) in (VP).*

Proof Let $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ be a feasible solution in (G-VMWD2) such that $\overline{y} \in D$. By means of contradiction, we suppose that there exists $\widetilde{x} \in D$ such that

$$f(\widetilde{x}) < f(\overline{y}).$$

By assumption, f is G_f-invex with respect to η at $\overline{y}$ on $D \cup pr_X W_2$. Then, from the definition of G_f-invexity, G_{f_i}, $i \in I$, are strictly increasing functions on $I_{f_i}(X)$. Thus,

$$G_{f_i}(f_i(\widetilde{x})) < G_{f_i}(f_i(\overline{y})), \quad i \in I. \tag{36}$$

Since $\overline{\lambda} \geq 0$, then (36) gives

$$\sum_{i=1}^{k} \overline{\lambda}_i G_{f_i}(f(\widetilde{x})) < \sum_{i=1}^{k} \overline{\lambda}_i G_{f_i}(f(\overline{y})). \tag{37}$$

By assumption, $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ is feasible in (G-VMWD2). Hence,

$$\sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\widetilde{x})\right) \leqq \sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\overline{y})\right). \tag{38}$$

Since $\overline{y} \in D$ and $\widetilde{x} \in D$, therefore,

$$\sum_{t=1}^{p} \overline{\mu}_t G_{h_t}(h_t(\widetilde{x})) - \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}(h_t(\overline{y})) = 0. \tag{39}$$

By assumption, f is G_f-invex with respect to η at $\overline{y}$ on $D \cup pr_X W_2$, g is G_g-invex with respect to η at $\overline{y}$ on $D \cup pr_X W_2$, h_t, $t \in T^+(\overline{y})$, is G_{h_t}-invex with respect to η at $\overline{y}$ on $D \cup pr_X W_2$, and h_t, $t \in T^-(\overline{y})$, is G_{h_t}-incave with respect to η at $\overline{y}$ on $D \cup pr_X W_2$. Then by Definition 2, we have, respectively,

$$G_{f_i}(f_i(\widetilde{x})) - G_{f_i}(f_i(y)) - G'_{f_i}(f_i(\overline{y}))\nabla f_i(\overline{y})\eta(\widetilde{x}, \overline{y}) \geqq 0, \quad i \in I, \tag{40}$$

$$G_{g_j}\left(g_j(\widetilde{x})\right) - G_{g_j}\left(g_j(\overline{y})\right) \geqq G'_{g_j}\left(g_j(\overline{y})\right)\nabla g_j(\overline{y})\,\eta(\widetilde{x}, \overline{y}), \quad j \in J, \tag{41}$$

$$G_{h_t}\left(h_t(\widetilde{x})\right) - G_{h_t}\left(h_t(\overline{y})\right) \geqq G'_{h_t}\left(h_t(\overline{y})\right) \nabla h_t(\overline{y}) \eta(\widetilde{x}, \overline{y}), \quad t \in T^+(\overline{y}), \tag{42}$$

$$G_{h_t}\left(h_t(\widetilde{x})\right) - G_{h_t}\left(h_t(\overline{y})\right) \leqq G'_{h_t}\left(h_t(\overline{y})\right) \nabla h_t(\overline{y}) \eta(\widetilde{x}, \overline{y}), \quad t \in T^-(\overline{y}). \tag{43}$$

By (37)–(39), we obtain

$$\sum_{i=1}^{k} \overline{\lambda}_i G'_{f_i}\left(f_i(\overline{y})\right) \nabla f_i(\overline{y}) \eta(\widetilde{x}, \overline{y}) < 0,$$

$$\sum_{j=1}^{m} \overline{\xi}_j G'_{g_j}\left(g_j(\overline{y})\right) \nabla g_j(\overline{y}) \eta(\widetilde{x}, \overline{y}) \leqq 0,$$

$$\sum_{t=1}^{p} \overline{\mu}_t G'_{h_t}\left(h_t(\overline{y})\right) \nabla h_t(\overline{y}) \eta(\widetilde{x}, \overline{y}) \leqq 0.$$

Adding both sides of inequalities above, we get

$$\left[\sum_{i=1}^{k} \overline{\lambda}_i G'_{f_i}\left(f_i(\overline{y})\right) \nabla f_i(\overline{y}) + \sum_{j=1}^{m} \overline{\xi}_j G'_{g_j}\left(g_j(\overline{y})\right) \nabla g_j(\overline{y}) + \sum_{t=1}^{p} \overline{\mu}_t G'_{h_t}\left(h_t(\overline{y})\right) \nabla h_t(\overline{y})\right] \eta(\widetilde{x}, \overline{y}) < 0,$$

contradicting the feasibility of $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ in (G-VMWD2).

Proof for $\overline{y}$ to be Pareto optimal in (VP) is similar, but f has to be assumed strictly G_f-invex with respect to η at $\overline{y}$ on $D \cup pr_X W_2$. □

Remark 25 We have proved two converse duality theorems for the introduced vector dual problem (G-VMWD2). Note that one of them, called the no-maximal G-converse duality theorem (Theorem 24), has been established without assuming that the feasible solution $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ is a (weak) maximum in (G-VMWD2) as it is assumed in various standard converse duality theorems in the literature. Furthermore, as follows from the proof of Theorem 24, we have proved it also without assuming some extra conditions imposed on the functions G_{g_j}, $j \in J$ and G_{h_t}, $t \in T$. Therefore, it is not difficult to see that some weaker hypotheses have been assumed to prove the introduced no-maximal G-converse duality theorems than various standard converse duality theorems known in the literature.

Theorem 26 (G-restricted converse duality) *Let $\overline{x}$ and $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ be feasible solutions in (VP) and (G-VMWD2), respectively, such that*

$$G_{f_i}\left(f_i(\overline{x})\right) = G_{f_i}\left(f_i(\overline{y})\right).$$

Moreover, assume that f is G_f-invex with respect to η at $\overline{y}$ on $D \cup pr_X W_2$, g is G_g-invex with respect to η at $\overline{y}$ on $D \cup pr_X W_2$, h_t, $t \in T^+(\overline{y})$, is G_{h_t}-invex with respect to η at $\overline{y}$ on $D \cup pr_X W_2$, and h_t, $t \in T^-(\overline{y})$, is G_{h_t}-incave with respect to η at $\overline{y}$ on $D \cup pr_X W_2$. Then $\overline{x}$ is weak Pareto optimal in (VP) and $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ is a weak maximum in (G-VMWD2).

Proof This follows on the line of the proof of Theorem 21. □

5 *G*-Wolfe duality

To prove some G-Wolfe duality results for the considered multiobjective programming problem (VP), we define the so-called vector G-Lagrange function $L_G : X \times R^k \times R^m \times R^p \to R^k$ defined by

$$L_G(y, \lambda, \xi, \mu) = diag\,\lambda \left(G_{f_1}(f_1(y)), \ldots, G_{f_k}(f_k(y))\right)^T + \sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(y)\right) \mathbf{e} + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(y))\, \mathbf{e}$$

$$= \left(\lambda_1 G_{f_1}(f_1(y)) + \sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(y)\right) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(y)), \ldots, \lambda_k G_{f_k}(f_k(y)) + \sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(y)\right) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(y))\right)$$

where

$$diag\,\lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \lambda_k \end{bmatrix}.$$

Relative to problem (VP), we consider the following vector G-Wolfe dual:

$$\left(G_{f_1}(f_1(y)) + \sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(y)\right) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(y)), \ldots, G_{f_k}(f_k(y)) + \sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(y)\right) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(y))\right) \to \max$$

$$\left[\sum_{i=1}^{k} \lambda_i G'_{f_i}(f_i(y))\, \nabla f_i(y) + \sum_{j=1}^{m} \xi_j G'_{g_j}\left(g_j(y)\right) \nabla g_j(y) + \sum_{t=1}^{p} \mu_t G'_{h_t}(h_t(y))\, \nabla h_t(y)\right] \eta(x, y) \geqq 0, \quad \forall x \in D, \qquad (G\text{-VWD})$$

$$y \in X,$$

$$\lambda \in R^k,\ \lambda \geq 0,\ \lambda \mathbf{e} = 1,$$

$$\xi \in R^m,\ \xi \geqq 0,$$

$$\mu \in R^q,$$

where G_{f_i}, $i \in I$, *are differentiable real-valued strictly increasing functions defined on* $I_{f_i}(X)$, G_{g_j}, $j \in J$, *are differentiable real-valued strictly increasing functions defined on*

$I_{g_j}(X)$, and G_{h_t}, $t \in T$, are differentiable real-valued strictly increasing functions defined on $I_{h_t}(X)$.

Let $\widetilde{W}$ denote the set of all feasible solutions in (G-VWD) and $pr_X \widetilde{W}$ be the projection of the set $\widetilde{W}$ on X, that is, $pr_X \widetilde{W} := \{y \in X : (y, \lambda, \xi, \mu) \in \widetilde{W}\}$.

Theorem 27 (G-weak duality) *Let x and (y, λ, ξ, μ) be any feasible solutions for (VP) and (G-VWD), respectively. Further, assume that f is G_f-invex with respect to η at $y \in pr_X \widetilde{W}$ on $D \cup pr_X \widetilde{W}$, g is G_g-invex with respect to η at $y \in pr_X \widetilde{W}$ on $D \cup pr_X \widetilde{W}$, h_t, $t \in T^+(y)$, is G_{h_t}-invex with respect to η at $y \in pr_X \widetilde{W}$ on $D \cup pr_X \widetilde{W}$, and h_t, $t \in T^-(y)$, is G_{h_t}-incave with respect to η at $y \in pr_X \widetilde{W}$ on $D \cup pr_X \widetilde{W}$, $G_{g_j}(0) = 0$ for $j \in J$ and $G_{h_t}(0) = 0$ for $t \in T^+(y) \cup T^-(y)$. Then*

$$\left(G_{f_1}(f_1(x)), \ldots, G_{f_k}(f_k(x))\right) \nless \left(G_{f_1}(f_1(y)) + \sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(y)\right) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(y)), \ldots, G_{f_k}(f_k(y)) + \sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(y)\right) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(y))\right). \tag{44}$$

Proof Let x and (y, λ, ξ, μ) be any feasible solutions for (VP) and (G-VWD), respectively. We proceed by, contradiction. Suppose that

$$\left(G_{f_1}(f_1(x)), \ldots, G_{f_k}(f_k(x))\right) < \left(G_{f_1}(f_1(y)) + \sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(y)\right) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(y)), \ldots, G_{f_k}(f_k(y)) + \sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(y)\right) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(y))\right). \tag{45}$$

Therefore, for any $i \in I$,

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(y)) < \sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(y)\right) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(y)). \tag{46}$$

Since $\lambda \geq 0$, then (46) gives

$$\sum_{i=1}^{k} \lambda_i G_{f_i}(f_i(x)) - \sum_{i=1}^{k} \lambda_i G_{f_i}(f_i(y)) < \sum_{i=1}^{k} \lambda_i \left[\sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(y)\right) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(y))\right].$$

From the feasibility of (y, λ, ξ, μ) in (G-VWD), we have $\sum_{i=1}^{k} \lambda_i = 1$. Then, the inequality above implies

$$\sum_{i=1}^{k} \lambda_i G_{f_i}(f_i(x)) - \sum_{i=1}^{k} \lambda_i G_{f_i}(f_i(y)) < \sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(y)\right) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(y)). \tag{47}$$

From $x \in D$ follows that $g_j(x) \leqq 0$, $j \in J$ and $h_t(x) = 0$, $t \in T$. Since G_{g_j} for $j \in J$ and G_{h_t} for $t \in T$ are strictly increasing functions on their domains, then

$$G_{g_j}\left(g_j(x)\right) \leqq G_{g_j}(0), \quad j \in J, \tag{48}$$

$$G_{h_t}(h_t(x)) = G_{h_t}(0), \quad t \in T. \tag{49}$$

By assumption, $G_{g_j}(0) = 0$ for $j \in J$ and $G_{h_t}(0) = 0$ for $t \in T^+(y) \cup T^-(y)$. Therefore, (48) and (49) yield, respectively,

$$G_{g_j}\left(g_j(x)\right) \leqq 0, \quad j \in J,$$

$$G_{h_t}(h_t(x)) = 0, \quad t \in T^+(y) \cup T^-(y).$$

Thus, from the feasibility of (y, λ, ξ, μ) in (G-VWD), it follows that

$$\sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(x)\right) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(x)) \leqq 0. \tag{50}$$

By assumption, f is G_f-invex with respect to η at $y \in pr_X\widetilde{W}$ on $D \cup pr_X\widetilde{W}$, g is G_g-invex with respect to η at $y \in pr_X\widetilde{W}$ on $D \cup pr_X\widetilde{W}$, $h_t, t \in T^+(y)$, is G_{h_t}-invex with respect to η at $y \in pr_X\widetilde{W}$ on $D \cup pr_X\widetilde{W}$, and h_t, $t \in T^-(y)$, is G_{h_t}-incave with respect to η at $y \in pr_X\widetilde{W}$ on $D \cup pr_X\widetilde{W}$. Then, by Definition 2, we have, respectively,

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(y)) - G'_{f_i}(f_i(y))\nabla f_i(y)\eta(x, y) \geqq 0, \quad i \in I,$$

$$G_{g_j}\left(g_j(x)\right) - G_{g_j}\left(g_j(y)\right) \geqq G'_{g_j}\left(g_j(y)\right)\nabla g_j(y)\,\eta(x, y), \quad j \in J,$$

$$G_{h_t}(h_t(x)) - G_{h_t}(h_t(y)) \geqq G'_{h_t}(h_t(y))\nabla h_t(y)\eta(x, y), \quad t \in T^+(y),$$

$$G_{h_t}(h_t(x)) - G_{h_t}(h_t(y)) \geqq G'_{h_t}(h_t(y))\nabla h_t(y)\eta(x, y), \quad t \in T^-(y).$$

From the feasibility of (y, λ, ξ, μ) in (G-VWD), it follows that

$$\sum_{i=1}^{k} \lambda_i G_{f_i}(f_i(x)) - \sum_{i=1}^{k} \lambda_i G_{f_i}(f_i(y)) \geqq \sum_{i=1}^{k} \lambda_i G'_{f_i}(f_i(y))\nabla f_i(y)\eta(x, y) \tag{51}$$

$$\sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(x)\right) - \sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(y)\right) \geqq \sum_{j=1}^{m} \xi_j G'_{g_j}\left(g_j(y)\right)\nabla g_j(y)\,\eta(x, y), \tag{52}$$

$$\sum_{t=1}^{p} \mu_t G_{h_t}(h_t(x)) - \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(y)) \geqq \sum_{t=1}^{p} \mu_t G'_{h_t}(h_t(y))\nabla h_t(y)\eta(x, y). \tag{53}$$

By (47) and (51),

$$\sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(y)\right) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(y)) > \sum_{i=1}^{k} \lambda_i G'_{f_i}(f_i(y))\nabla f_i(y)\eta(x, y). \tag{54}$$

Adding both sides of inequalities (52)–(54), we get

$$\sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(x)\right) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(x)) \geqq \left[\sum_{i=1}^{k} \lambda_i G'_{f_i}(f_i(y))\nabla f_i(y) + \sum_{j=1}^{m} \xi_j G'_{g_j}\left(g_j(y)\right)\nabla g_j(y) + \sum_{t=1}^{p} \mu_t G'_{h_t}(h_t(y))\nabla h_t(y)\right]\eta(x, y)$$

Using (50), we obtain the following inequality

$$\left[\sum_{i=1}^{k}\lambda_i G'_{f_i}\left(f_i(y)\right)\nabla f_i(y)+\sum_{j=1}^{m}\xi_j G'_{g_j}\left(g_j(y)\right)\nabla g_j(y)\right.$$
$$\left.+\sum_{i=1}^{k}\lambda_i G'_{f_i}\left(f_i(y)\right)\nabla f_i(y)\right]\eta(x,y)<0$$

contradicting the feasibility of (y, λ, ξ, μ) in (G-VWD). □

Now, we prove G-weak duality theorem under invexity assumption imposed on the G-Lagrange function L_G.

Theorem 28 (G-weak duality) *Let x and (y, λ, ξ, μ) be any feasible solutions for (VP) and (G-VWD), respectively. Further, assume that the G-Lagrange function L_G is invex with respect to η at $y \in pr_X\widetilde{W}$ on $D \cup pr_X\widetilde{W}$, $G_{g_j}(0) = 0$ for $j \in J$ and $G_{h_t}(0) = 0$ for $t \in T^+(y) \cup T^-(y)$. Then*

$$\left(G_{f_1}\left(f_1(x)\right),\ldots,G_{f_k}\left(f_k(x)\right)\right)\nless\left(G_{f_1}\left(f_1(y)\right)+\sum_{j=1}^{m}\xi_j G_{g_j}\left(g_j(y)\right)\right.$$
$$\left.+\sum_{t=1}^{p}\mu_t G_{h_t}\left(h_t(y)\right),\ldots,G_{f_k}\left(f_k(y)\right)+\sum_{j=1}^{m}\xi_j G_{g_j}\left(g_j(y)\right)+\sum_{t=1}^{p}\mu_t G_{h_t}\left(h_t(y)\right)\right)$$

Proof Let x and (y, λ, ξ, μ) be any feasible solutions for (VP) and (G-VWD), respectively. We proceed by contradiction. Suppose that

$$\left(G_{f_1}\left(f_1(x)\right),\ldots,G_{f_k}\left(f_k(x)\right)\right)<\left(G_{f_1}\left(f_1(y)\right)+\sum_{j=1}^{m}\xi_j G_{g_j}\left(g_j(y)\right)\right.$$
$$\left.+\sum_{t=1}^{p}\mu_t G_{h_t}\left(h_t(y)\right),\ldots,G_{f_k}\left(f_k(y)\right)+\sum_{j=1}^{m}\xi_j G_{g_j}\left(g_j(y)\right)+\sum_{t=1}^{p}\mu_t G_{h_t}\left(h_t(y)\right)\right).$$

In the similar manner as in the proof of Theorem 27, we obtain the inequality (47). By assumption, $G_{g_j}(0) = 0$ for $j \in J$ and $G_{h_t}(0) = 0$ for $t \in T^+(y) \cup T^-(y)$. Then, from the feasibility of x in (VP) and (y, λ, ξ, μ) in (G-VWD) follows that

$$\sum_{j=1}^{m}\xi_j G_{g_j}\left(g_j(x)\right)+\sum_{t=1}^{p}\mu_t G_{h_t}\left(h_t(x)\right)\leqq 0. \tag{55}$$

By (47) and (55), we get

$$\sum_{i=1}^{k}\lambda_i G_{f_i}\left(f_i(x)\right)+\sum_{j=1}^{m}\xi_j G_{g_j}\left(g_j(x)\right)+\sum_{t=1}^{p}\mu_t G_{h_t}\left(h_t(x)\right)$$
$$<\sum_{i=1}^{k}\lambda_i G_{f_i}\left(f_i(y)\right)+\sum_{j=1}^{m}\xi_j G_{g_j}\left(g_j(y)\right)+\sum_{t=1}^{p}\mu_t G_{h_t}\left(h_t(y)\right). \tag{56}$$

By assumption, the G-Lagrange function L_G is invex with respect to η at $y \in pr_X \widetilde{W}$ on $D \cup pr_X \widetilde{W}$. Then, by Remark 4, it follows that

$$L_G(x, \lambda, \xi, \mu) - L_G(y, \lambda, \xi, \mu) \geqq \nabla L_G(y, \lambda, \xi, \mu)\, \eta(x, y).$$

Hence, from the definition of the G-Lagrange function, it follows for any $i = 1, \ldots, k$,

$$\begin{aligned}
&\lambda_i G_{f_i}(f_i(x)) + \left[\sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(x)\right) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(x))\right] \\
&-\lambda_i G_{f_i}(f_i(y)) - \left[\sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(y)\right) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(y))\right] \\
&\geqq \left[\sum_{i=1}^{k} \lambda_i G'_{f_i}(f_i(y)) \nabla f_i(y) + \sum_{j=1}^{m} \xi_j G'_{g_j}\left(g_j(y)\right) \nabla g_j(y)\right. \\
&\quad \left. + \sum_{t=1}^{p} \mu_t G'_{h_t}(h_t(y)) \nabla h_t(y)\right] \eta(x, y).
\end{aligned}$$

Adding both sides of the inequalities above and using $\sum_{i=1}^{k} \lambda_i = 1$, we get

$$\begin{aligned}
&\sum_{i=1}^{k} \lambda_i G_{f_i}(f_i(x)) + \left[\sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(x)\right) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(x))\right] \\
&\quad - \left(\sum_{i=1}^{k} \lambda_i G_{f_i}(f_i(y)) + \left[\sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(y)\right) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(y))\right]\right) \\
&\geqq \left[\sum_{i=1}^{k} \lambda_i G'_{f_i}(f_i(y)) \nabla f_i(y) + \sum_{j=1}^{m} \xi_j G'_{g_j}\left(g_j(y)\right) \nabla g_j(y)\right. \\
&\quad \left. + \sum_{t=1}^{p} \mu_t G'_{h_t}(h_t(y)) \nabla h_t(y)\right] \eta(x, y).
\end{aligned} \tag{57}$$

By (56) and (57), it follows that the following inequality

$$\left[\sum_{i=1}^{k} \lambda_i G'_{f_i}(f_i(y)) \nabla f_i(y) + \sum_{j=1}^{m} \xi_j G'_{g_j}\left(g_j(y)\right) \nabla g_j(y) + \sum_{t=1}^{p} \mu_t G'_{h_t}(h_t(y)) \nabla h_t(y)\right] \eta(x, y) < 0$$

holds. But this is a contradiction to the feasibility of (y, λ, ξ, μ) in (G-VWD). □

Theorem 29 (G-strong duality) *Let $\overline{x}$ be a (weak) Pareto optimal solution in (VP) and the Kuhn–Tucker constraint qualification be satisfied at $\overline{x}$. Then there exist $\overline{\lambda} \in R^k_+$, $\overline{\xi} \in R^m_+$, $\overline{\mu} \in R^q$ such that $\left(\overline{x}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ is feasible for (G-VWD) and the objective functions of (VP) and (G-VWD) are equal at these points. If also G-weak duality (Theorem 27) between (VP) and (G-VWD) holds, then $\left(\overline{x}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ is a (weak) maximum point in (G-VWD).*

Theorem 30 (G-converse duality) *Let $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ be a (weak) maximum in (G-VWD) and $\overline{y} \in D$. Moreover, assume that the (strictly) G-Lagrange function is invex with respect η at $\overline{y}$ on $D \cup pr_X \widetilde{W}$. Then $\overline{y}$ is (weak) Pareto optimal in (VP).*

Proof Let $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ be a (weak) maximum in (G-VWD) such that $\overline{y} \in D$. Suppose, contrary to the result, that $\overline{y}$ is not a weak Pareto optimal point for (VP), that is, there exists $\widetilde{x} \in D$ such that

$$f\left(\widetilde{x}\right) < f\left(\overline{y}\right).$$

From the definition G_{f_i}, $i \in I$, is a strictly increasing function on $I_{f_i}(X)$. Thus,

$$G_{f_i}\left(f_i\left(\widetilde{x}\right)\right) < G_{f_i}\left(f_i\left(\overline{y}\right)\right). \tag{58}$$

By assumption, $\left(\overline{y}, \lambda, \overline{\xi}, \overline{\mu}\right)$ is a weak maximum in (G-VWD). Hence, by the G-Karush–Kuhn–Tucker necessary optimality condition (3),

$$\sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\widetilde{x})\right) \leqq \sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\overline{y})\right). \tag{59}$$

Since $\widetilde{x} \in D$ and $\overline{y} \in D$, then

$$\sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\widetilde{x})\right) - \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\overline{y})\right) = 0. \tag{60}$$

By (58)–(60), we get for any $i = 1, \ldots, k$,

$$\begin{aligned} &\left(G_{f_1}\left(f_1(\widetilde{x})\right), \ldots, G_{f_k}\left(f_k(\widetilde{x})\right)\right) + \left[\sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\widetilde{x})\right) + \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\widetilde{x})\right)\right] \mathbf{e} \\ &\quad < \left(G_{f_1}\left(f_1(\overline{y})\right), \ldots, G_{f_k}\left(f_k(\overline{y})\right)\right) + \left[\sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\overline{y})\right) + \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\overline{y})\right)\right] \mathbf{e}. \end{aligned} \tag{61}$$

Since $\overline{\lambda}_i \geq 0$, $i \in I$, then (61) yields

$$\begin{aligned} &\sum_{i=1}^{k} \overline{\lambda}_i G_{f_i}\left(f_i(\widetilde{x})\right) + \sum_{i=1}^{k} \overline{\lambda}_i \left[\sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\widetilde{x})\right) + \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\widetilde{x})\right)\right] \\ &\quad < \sum_{i=1}^{k} \overline{\lambda}_i G_{f_i}\left(f_i(\overline{y})\right) + \sum_{i=1}^{k} \overline{\lambda}_i \left[\sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\overline{y})\right) + \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\overline{y})\right)\right]. \end{aligned}$$

From the feasibility of $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ in (G-VWD), we have $\sum_{i=1}^{k} \overline{\lambda}_i = 1$. Then, the inequality above implies

$$\begin{aligned} &\sum_{i=1}^{k} \overline{\lambda}_i G_{f_i}\left(f_i(\widetilde{x})\right) + \left[\sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\widetilde{x})\right) + \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\widetilde{x})\right)\right] \\ &\quad < \sum_{i=1}^{k} \overline{\lambda}_i G_{f_i}\left(f_i(\overline{y})\right) + \left[\sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\overline{y})\right) + \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\overline{y})\right)\right]. \end{aligned} \tag{62}$$

By assumption, the G-Lagrange function L_G is invex with respect to η at $\overline{y} \in pr_X\widetilde{W}$ on $D \cup pr_X\widetilde{W}$. Then, by Remark 4, it follows that

$$L_G\left(\widetilde{x}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right) - L_G\left(\overline{y}, \overline{\mu}, \overline{\xi}, \overline{\mu}\right) \geqq \nabla L_G\left(\overline{y}, \overline{\mu}, \overline{\xi}, \overline{\mu}\right) \eta\left(\widetilde{x}, \overline{y}\right).$$

Hence, from the definition of the G-Lagrange function, it follows for any $i = 1, \ldots, k$,

$$\overline{\lambda}_i G_{f_i}\left(f_i\left(\widetilde{x}\right)\right) + \left[\sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\widetilde{x})\right) + \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\widetilde{x})\right)\right] - \Bigg(\overline{\lambda}_i G_{f_i}\left(f_i\left(\overline{y}\right)\right)$$
$$+ \left[\sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\overline{y})\right) + \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\overline{y})\right)\right]\Bigg) \geq \Bigg[\overline{\lambda}_i G'_{f_i}\left(f_i\left(\overline{y}\right)\right) \nabla f_i\left(\overline{y}\right)$$
$$+ \sum_{j=1}^{m} \overline{\xi}_j G'_{g_j}\left(g_j(\overline{y})\right) \nabla g_j(\overline{y}) + \sum_{t=1}^{p} \overline{\mu}_t G'_{h_t}\left(h_t(\overline{y})\right) \nabla h_t(\overline{y})\Bigg] \eta\left(\widetilde{x}, \overline{y}\right)$$

Adding both sides of the inequalities above and using $\sum_{i=1}^{k} \overline{\lambda}_i = 1$, we get

$$\sum_{i=1}^{k} \overline{\lambda}_i G_{f_i}\left(f_i\left(\widetilde{x}\right)\right) + \left[\sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\widetilde{x})\right) + \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\widetilde{x})\right)\right] - \sum_{i=1}^{k} \overline{\lambda}_i G_{f_i}\left(f_i\left(\overline{y}\right)\right)$$
$$- \left[\sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\overline{y})\right) + \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\overline{y})\right)\right] \geq \Bigg[\sum_{i=1}^{k} \overline{\lambda}_i G'_{f_i}\left(f_i\left(\overline{y}\right)\right) \nabla f_i\left(\overline{y}\right)$$
$$+ \sum_{j=1}^{m} \overline{\xi}_j G'_{g_j}\left(g_j(\overline{y})\right) \nabla g_j(\overline{y}) + \sum_{t=1}^{p} \overline{\mu}_t G'_{h_t}\left(h_t(\overline{y})\right) \nabla h_t(\overline{y})\Bigg] \eta\left(\widetilde{x}, \overline{y}\right). \tag{63}$$

By (62) and (63), we obtain the following inequality

$$\left[\sum_{i=1}^{k} \overline{\lambda}_i G'_{f_i}\left(f_i\left(\overline{y}\right)\right) \nabla f_i\left(\overline{y}\right) + \sum_{j=1}^{m} \overline{\xi}_j G'_{g_j}\left(g_j(\overline{y})\right) \nabla g_j(\overline{y}) + \sum_{t=1}^{p} \overline{\mu}_t G'_{h_t}\left(h_t(\overline{y})\right) \nabla h_t(\overline{y})\right]$$
$$\eta\left(\widetilde{x}, \overline{y}\right) < 0$$

which contradicts the dual constraint of problem (G-VWD). Thus, the conclusion of theorem is proved. □

Theorem 31 (G-restricted converse duality) *Let $\overline{x}$ and $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ be feasible solutions in (VP) and (G-VWD), respectively, such that*

$$\left(G_{f_1}\left(f_1(\overline{x})\right), \ldots, G_{f_k}\left(f_k(\overline{x})\right)\right) = \left(G_{f_1}\left(f_1(\overline{y})\right), \ldots, G_{f_k}\left(f_k(\overline{y})\right)\right)$$
$$+ \left[\sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j\left(\overline{y}\right)\right) + \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\overline{y})\right)\right] \mathbf{e}.$$

Moreover, assume that, for any fixed $\bar{\lambda} \in R^k$, $\bar{\lambda} \geq 0$, $\bar{\xi} \in R^m$, $\bar{\xi} \geqq 0$, $\bar{\mu} \in R^p$, the G-Lagrange function is (invex) strictly invex at $\overline{y}$ on $D \cup pr_X\widetilde{W}$ with respect to η. Then $\overline{x}$ is (weak Pareto optimal) Pareto optimal in (VP) and $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ is a (weak maximum) maximum in (G-VWD).

Proof Follows directly from the weak duality theorem (Theorem 28). □

6 *G*-mixed vector duality

In this section, we introduce two new vector dual problems for the considered multiobjective programming problem (VP). We will call them G-mixed vector dual problems for the multiobjective programming problem (VP).

Now, relative to the primal multiobjective programming problem (VP), we consider the following vector dual problem:

$$\begin{aligned}
&\Bigg(G_{f_1}\left(f_1(y)\right)+\sum_{j=1}^{m}\xi_j G_{g_j}\left(g_j(y)\right)+\sum_{t=1}^{p}\mu_t G_{h_t}\left(h_t(y)\right),\ldots,\\
&G_{f_k}\left(f_k(y)\right)+\sum_{j=1}^{m}\xi_j G_{g_j}\left(g_j(y)\right)+\sum_{t=1}^{p}\mu_t G_{h_t}\left(h_t(y)\right)\Bigg)\to\max\\
&\Bigg[\sum_{i=1}^{k}\lambda_i G'_{f_i}\left(f_i(y)\right)\nabla f_i\left(y\right)+\sum_{j=1}^{m}\xi_j G'_{g_j}\left(g_j(y)\right)\nabla g_j\left(y\right)\\
&\qquad+\sum_{t=1}^{p}\mu_t G'_{h_t}\left(h_t(y)\right)\nabla h_t(y)\Bigg]\eta\left(x,y\right)\geqq 0,\ \forall x\in D,\\
&\sum_{j=1}^{m}\xi_j G_{g_j}\left(g_j\left(y\right)\right)+\sum_{t=1}^{p}\mu_t G_{h_t}\left(h_t(y)\right)\geqq 0, \qquad (G\text{-VMD1})\\
&y\in X,\\
&\lambda\in R^k,\lambda\geq 0,\ \lambda^T\mathbf{e}=1,\\
&\xi\in R^m,\ \xi\geqq 0,\\
&\mu\in R^q,
\end{aligned}$$

where G_{f_i}, $i\in I$, are differentiable real-valued strictly increasing functions defined on $I_{f_i}(X)$, G_{g_j}, $j\in J$, are differentiable real-valued strictly increasing functions defined on $I_{g_j}(X)$, and G_{h_t}, $t\in T$, are differentiable real-valued strictly increasing functions defined on $I_{h_t}(X)$. Since the set of all feasible solutions for problem (G-VMD1) is the same as the set of all feasible solutions for problem (G-VMWD1), we denote it by W_1.

Theorems 32–35 contain some results for the vector dual problem (G-VMD1). Their proofs are similar to the proofs of Theorems 27–30 and therefore are omitted.

Theorem 32 (G-weak mixed duality) *Let x and (y,λ,ξ,μ) be any feasible solutions for (VP) and (G-VMD1), respectively. Further, assume that f is G_f-invex with respect to η at $y\in pr_X W_1$ on $D\cup pr_X W_1$, g is G_g-invex with respect to η at $y\in pr_X W_1$ on $D\cup pr_X W_1$, h_t, $t\in T^+(y)$, is G_{h_t}-invex with respect to η at $y\in pr_X W_1$ on $D\cup pr_X W_1$, and h_t, $t\in T^-(y)$, is G_{h_t}-incave with respect to η at $y\in pr_X W_1$ on $D\cup pr_X W_1$, $G_{g_j}(0)=0$ for $j\in J$ and $G_{h_t}(0)=0$ for $t\in T^+(y)\cup T^-(y)$. Then*

$$\left(G_{f_1}\left(f_1\left(x\right)\right),\ldots,G_{f_k}\left(f_k\left(x\right)\right)\right) \nless \left(G_{f_1}\left(f_1(y)\right)+\sum_{j=1}^{m}\xi_j G_{g_j}\left(g_j(y)\right)\right.$$
$$\left.+\sum_{t=1}^{p}\mu_t G_{h_t}\left(h_t(y)\right),\ldots,G_{f_k}\left(f_k(y)\right)+\sum_{j=1}^{m}\xi_j G_{g_j}\left(g_j(y)\right)+\sum_{t=1}^{p}\mu_t G_{h_t}\left(h_t(y)\right)\right). \tag{64}$$

Theorem 33 (G-weak mixed duality) *Let x and (y,λ,ξ,μ) be any feasible points for (VP) and (G-VMD1), respectively. If the G-Lagrange function is invex with respect to η at y on $D\cup pr_X W_1$, $G_{g_j}(0)=0$ for $j\in J$ and $G_{h_t}(0)=0$ for $t\in T^+(y)\cup T^-(y)$, then the relation (64) is fulfilled.*

Theorem 34 (G-strong mixed duality) *Let $\overline{x}$ be a (weak) Pareto optimal in (VP) and the Kuhn–Tucker constraint qualification be satisfied at $\overline{x}$. Then there exist $\overline{\lambda}\in R^k_+$, $\overline{\xi}\in R^m_+$, $\overline{\mu}\in R^p$ such that $\left(\overline{x},\overline{\lambda},\overline{\xi},\overline{\mu}\right)$ is feasible for (G-VMD1) and the objective functions of (VP) and (G-VMD1) are equal at these points. If also G-weak duality (Theorem 32) between (VP) and (G-VMD1) holds, then $\left(\overline{x},\overline{\lambda},\overline{\xi},\overline{\mu}\right)$ is a (weak) maximum point for (G-VMD1).*

Theorem 35 (G-converse mixed duality) *Let $\left(\overline{y},\overline{\lambda},\overline{\xi},\overline{\mu}\right)$ be a (weak) maximum for (G-VMD1) such that $\overline{y}\in D$. Moreover, we assume that the G-Lagrange function is invex with respect η at $\overline{y}$ on $D\cup pr_X W_1$. Then $\overline{y}$ is (weak) Pareto optimal in (VP).*

Theorem 36 (No-maximal G-converse mixed duality) *Let $\left(\overline{y},\overline{\lambda},\overline{\xi},\overline{\mu}\right)$ be a feasible solution for (G-VMD1) such that $\overline{y}\in D$. Moreover, we assume that the G-Lagrange function is (invex) strictly invex with respect η at $\overline{y}$ on $D\cup pr_X W_1$, $G_{g_j}(0)=0$ for $j\in J$ and $G_{h_t}(0)=0$ for $t\in T^+(\overline{y})\cup T^-(\overline{y})$. Then $\overline{y}$ is (weak) Pareto optimal in (VP).*

Proof Let $\left(\overline{y},\overline{\lambda},\overline{\xi},\overline{\mu}\right)$ be feasible in (G-VMD1) such that $\overline{y}\in D$. Suppose, contrary to the result, that $\overline{y}$ is not a weak Pareto optimal solution in (VP). Then there exists $\widetilde{x}\in D$ such that

$$f\left(\widetilde{x}\right) < f\left(\overline{y}\right).$$

From the definition of G-invexity, every function G_{f_i}, $i\in I$, is strictly increasing on its domain. Therefore, for any $i\in I$, the above inequality yields

$$G_{f_i}\left(f_i\left(\widetilde{x}\right)\right) < G_{f_i}\left(f_i\left(\overline{y}\right)\right).$$

Since $\overline{\lambda}\geq 0$, then the inequalities above imply

$$\sum_{i=1}^{k}\overline{\lambda}_i G_{f_i}\left(f_i(\widetilde{x})\right) < \sum_{i=1}^{k}\overline{\lambda}_i G_{f_i}\left(f_i(\overline{y})\right). \tag{65}$$

By assumption, the G-Lagrange function L_G is invex with respect to η at y on $D\cup pr_X W_1$. Thus,

$$L_G\left(\widetilde{x},\overline{\lambda},\overline{\xi},\overline{\mu}\right)-L_G\left(\overline{y},\overline{\lambda},\overline{\xi},\overline{\mu}\right)\geqq \nabla L_G\left(\overline{y},\overline{\lambda},\overline{\xi},\overline{\mu}\right)\eta\left(\widetilde{x},\overline{y}\right). \tag{66}$$

Since $\left(\overline{x}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ is feasible for (G-VMD1) then, using the definition of the G-Lagrange function L_G, we get, for any $i \in I$,

$$\overline{\lambda}_i G_{f_i}\left(f_i(\widetilde{x})\right) + \sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\widetilde{x})\right) + \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\widetilde{x})\right) - \overline{\lambda}_i G_{f_i}\left(f_i(\overline{y})\right)$$

$$\left[\sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\overline{y})\right) + \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\overline{y})\right)\right] \geqq \left[G'_{f_i}\left(f(\overline{y})\right) \nabla f_i(\overline{y})\right. \tag{67}$$

$$\left. + \sum_{j=1}^{m} \overline{\xi}_j G'_{g_j}\left(g_j(\overline{y})\right) \nabla g_j(\overline{y}) + \sum_{t=1}^{p} \overline{\mu}_t G'_{h_t}\left(h_t(\overline{y})\right) \nabla h_t(\overline{y})\right] \eta(\widetilde{x}, \overline{y})$$

Adding both sides of the above inequalities and using $\sum_{i=1}^{k} \overline{\lambda}_i = 1$, we obtain

$$\sum_{i=1}^{k} \overline{\lambda}_i G_{f_i}\left(f_i(\widetilde{x})\right) + \sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\widetilde{x})\right) + \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\widetilde{x})\right) - \left[\sum_{i=1}^{k} \overline{\lambda}_i G_{f_i}\left(f_i(\overline{y})\right)\right.$$

$$\left. + \sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\overline{y})\right) + \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\overline{y})\right)\right] \geqq \left[\sum_{i=1}^{k} G'_{f_i}\left(f(\overline{y})\right) \nabla f_i(\overline{y})\right.$$

$$\left. + \sum_{j=1}^{m} \overline{\xi}_j G'_{g_j}\left(g_j(\overline{y})\right) \nabla g_j(\overline{y}) + \sum_{t=1}^{p} \overline{\mu}_t G'_{h_t}\left(h_t(\overline{y})\right) \nabla h_t(\overline{y})\right] \eta(\widetilde{x}, \overline{y}) \tag{68}$$

By assumption, $G_{g_j}(0) = 0$ for $j \in J$ and $G_{h_t}(0) = 0$ for $t \in T^+(\overline{y}) \cup T^-(\overline{y})$. Moreover, G_{g_j}, $j \in J$ and G_{h_t} $t \in T^+(\overline{y}) \cup T^-(\overline{y})$. Therefore, from $\widetilde{x} \in D$, it follows that

$$\sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\widetilde{x})\right) + \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\widetilde{x})\right) \leqq 0. \tag{69}$$

Hence, from the feasibility of $\left(\overline{x}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ in (G-VMD1), we obtain

$$\sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\widetilde{x})\right) + \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\widetilde{x})\right) \leqq \sum_{j=1}^{m} \overline{\xi}_j G_{g_j}\left(g_j(\overline{y})\right) + \sum_{t=1}^{p} \overline{\mu}_t G_{h_t}\left(h_t(\overline{y})\right). \tag{70}$$

By (65), (68) and (70), it follows that the following inequality

$$\left[\sum_{i=1}^{k} \overline{\lambda}_i G'_{f_i}\left(f(\overline{y})\right) \nabla f_i(\overline{y}) + \sum_{j=1}^{m} \overline{\xi}_j G'_{g_j}\left(g_j(\overline{y})\right) \nabla g_j(\overline{y}) + \sum_{t=1}^{p} \overline{\mu}_t G'_{h_t}\left(h_t(\overline{y})\right) \nabla h_t(\overline{y})\right]$$
$$\eta(\widetilde{x}, \overline{y}) < 0$$

holds, which contradicts the feasibility of $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ in (G-VMWD1). □

Based on the vector dual problem (G-VMWD2) in the format of Mond–Weir, in relation to (VP), we construct the following multiobjective dual problem

$$
\begin{aligned}
&\Bigg(G_{f_1}\left(f_1(y)\right)+\sum_{j=1}^{m}\xi_j G_{g_j}\left(g_j(y)\right)+\sum_{t=1}^{p}\mu_t G_{h_t}\left(h_t(y)\right),\ldots,\\
&\quad G_{f_k}\left(f_k(y)\right)+\sum_{j=1}^{m}\xi_j G_{g_j}\left(g_j(y)\right)+\sum_{t=1}^{p}\mu_t G_{h_t}\left(h_t(y)\right)\Bigg)\to \max\\
&\Bigg[\sum_{i=1}^{k}\lambda_i G'_{f_i}\left(f_i(y)\right)\nabla f_i(y)+\sum_{j=1}^{m}\xi_j G'_{g_j}\left(g_j(y)\right)\nabla g_j(y)\\
&\quad +\sum_{t=1}^{p}\mu_t G'_{h_t}\left(h_t(y)\right)\nabla h_t(y)\Bigg]\eta(x,y)\geqq 0,\ \forall x\in D,\\
&\sum_{j=1}^{m}\xi_j G_{g_j}\left(g_j(y)\right)\geqq 0,\\
&\sum_{t=1}^{p}\mu_t G_{h_t}\left(h_t(y)\right)\geqq 0,\\
&y\in X,\\
&\lambda\in R^k,\lambda\geq 0,\ \lambda^T\mathbf{e}=1,\\
&\xi\in R^m,\ \xi\geqq 0,\\
&\mu\in R^q.
\end{aligned}
\tag{G-VMD2}
$$

Analogously as for the vector G-mixed dual problem (G-VMD1), the same G-duality theorems are true for the considered vector mixed dual problem (G-VMD2). Also proofs for corresponding duality results for the above multiobjective dual problem run on the same lines as the proofs of the Theorems 32–36. Therefore, they were also omitted in this work.

7 Conclusion

In this paper, we have introduced several vector dual problems for the considered differentiable multiobjective programming problem with both inequality and equality constraints. The so-called vector G-dual problem in the sense of Mond–Weir, the vector G-dual problem in the sense of Wolfe and the vector G-mixed dual problem presented in this work are different from vector dual problems known in the literature. Various duality theorems between the primal multiobjective problem and the introduced vector G-dual problems have been proved under the assumption that the functions constituting these vector optimization problems are vector G-invex functions with respect to the same function η and with respect to, not necessarily, the same function G or under the assumption that the so-called G-Lagrange function (also introduced in this paper) is invex. This paper extends entirely earlier works, in which duality results have been obtained for a multiobjective programming problem by applying convexity, generalized convexity and even invexity assumptions imposed on functions involved in a multiobjective programming problem (see, for example, [6,9,15,23,27]). Further, we have considered two types of converse duality for the introduced vector G-dual problems. One of them corresponds with the standard converse duality known in the literature. However, the second one, called no-maximal G-converse duality, is a new type of duality results for multiobjective programming problems. As it is known from the literature, to prove the standard converse duality theorems (and also the standard G-converse duality theorems considered in this paper), it is assumed that the considered feasible solution $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ is (weak) maximal in these vector dual problems and $\overline{y}$ belongs to the set of all feasible solutions in the primal multiobjective programming problem (in fact, it is sufficient to assume that $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ is a feasible solution satisfying the G-Karush–Kuhn–Tucker necessary optimality conditions). However, to prove the introduced no-maximal G-dual converse theorems, it is sufficient to assume that $\left(\overline{y}, \overline{\lambda}, \overline{\xi}, \overline{\mu}\right)$ is a feasible solution in the vector dual problem and $\overline{y} \in D$. Moreover, some stronger assumptions should be imposed on the functions G_{g_j}, $j \in J$ and G_{h_t}, $t \in T^+(\overline{y}) \cup T^-(\overline{y})$ associated with inequality and equality constraints constituting the primal multiobjective programming problem (VP). Indeed, we have assumed that these functions satisfy the following conditions $G_{g_j}(0) = 0$ for $j \in J$ and $G_{h_t}(0) = 0$ for $t \in T^+(\overline{y}) \cup T^-(\overline{y})$. Furthermore, it turns out that if we modify slightly constraints of vector G-dual problems in the sense of Mond–Weir and G-mixed dual problems, then we can establish these new converse duality theorems without assuming extra assumptions imposed on the functions G_{g_j}, $j \in J$ and G_{h_t}, $t \in T^+(\overline{y}) \cup T^-(\overline{y})$. Thus, it is not difficult to see that some stronger hypotheses should be assumed to prove standard converse duality theorems known in the literature than for proving the introduced no-maximal G-dual converse theorems. For example, if we transform the vector G-mixed dual problem (G-VMD1) to the following form

$$\left(G_{f_1}(f_1(y)) + \sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(y)\right) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(y)), \ldots, \right.$$

$$\left. G_{f_k}(f_k(y)) + \sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(y)\right) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(y))\right) \to \max$$

$$\left[\sum_{i=1}^{k} \lambda_i G'_{f_i}(f_i(y)) \nabla f_i(y) + \sum_{j=1}^{m} \xi_j G'_{g_j}\left(g_j(y)\right) \nabla g_j(y) \right.$$

$$\left. + \sum_{t=1}^{p} \mu_t G'_{h_t}(h_t(y)) \nabla h_t(y)\right] \eta(x, y) \geqq 0, \forall x \in D,$$

$$\sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(y)\right) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(y))$$

$$\geqq \sum_{j=1}^{m} \xi_j G_{g_j}\left(g_j(x)\right) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(x)), \quad \forall x \in D, \quad (G\text{-VMD3})$$

$$y \in X,$$

$$\lambda \in R^k, \lambda \geq 0, \ \lambda^T \mathbf{e} = 1,$$

$$\xi \in R^m, \ \xi \geqq 0,$$

$$\mu \in R^q,$$

then the no-maximal G-converse duality theorem for such a vector G-dual problem can be proved without assuming that $G_{g_j}(0) = 0$ for $j \in J$ and $G_{h_t}(0) = 0$ for $t \in T^+(\overline{y}) \cup T^-(\overline{y})$. As follows from this example, some weaker conditions imposed on the functions constituting the G-mixed dual problem with modified constraints are assumed to prove the no–maximal G-converse duality theorem than for the standard converse dual theorem. The same is valid for the other introduced vector G-dual problems in the format of Mond–Weir and vector G-mixed dual problems. Indeed, if the second constraints of these vector dual problems are modified in the same way (as it has been shown above for the vector G-mixed dual problem (G-VMD1)), then G-no-maximal converse duality theorems can be established without assuming some extra condition imposed on the functions G_{g_j}, $j \in J$ and G_{h_t}, $t \in T$ (see also proofs of weak and converse duality theorems for the vector G-dual problem in the format of Mond–Weir (G-VMWD2)).

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